

Minimal periods of semilinear evolution equations with Lipschitz nonlinearity

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Abstract

It is known that any periodic orbit of a Lipschitz ordinary differential equation $\dot{x} = f(x)$ must have period at least $2\pi/L$, where L is the Lipschitz constant of f . In this paper we prove a similar result for the semilinear evolution equation $du/dt = -Au + f(u)$: for each α with $0 \leq \alpha \leq 1/2$ there exists a constant K_α such that if L is the Lipschitz constant of f as a map from $D(A^\alpha)$ into H then any periodic orbit has period at least $K_\alpha L^{-1/(1-\alpha)}$. As a concrete application we recover a result of Kukavica giving a lower bound on the period for the 2d Navier-Stokes equations with periodic boundary conditions.

1 Introduction

Yorke [12] showed that any periodic orbit of an ordinary differential equation $\dot{x} = f(x)$ must have period at least $2\pi/L$, where L is the (global) Lipschitz constant of f , i.e.

$$|f(x) - f(y)| \leq L|x - y|.$$

As well as being interesting in its own right, this result is useful since it allows one to show that the conditions required by Takens' time-delay embedding theorem are satisfied provided that the time delay is taken sufficiently small (see Sauer, Yorke, & Casdagli [10] for a proof of this theorem in the ODE case).

Recently Robinson [8] has proved a version of the Takens embedding theorem valid for infinite-dimensional systems, and so a similar result guaranteeing a minimum period would be useful in this context, as well as once again being of independent interest.

Although there is no general framework that will encompass all possible PDEs, the semilinear evolution equations studied by Henry [4] are general enough to include reaction-diffusion equations and the Navier-Stokes equations. Here we prove that any periodic orbit of the equation

$$\frac{du}{dt} = -Au + f(u),$$

where A is a positive self-adjoint operator and f has Lipschitz constant L from $D(A^\alpha)$ into H for $0 \leq \alpha \leq 1/2$, must have period at least $K_\alpha T^{-1/(1-\alpha)}$, where K_α depends only on α .

Our argument is inspired in part by that of Kukavica [5], who exploited the time analyticity of solutions of the Navier-Stokes equations to show that there is a lower bound on the period of any periodic orbit, even for the three-dimensional case (where existence and uniqueness results are not available in general).

In Section 2 we give a simple proof of the ODE result, and in Section 3 we give the new result for semilinear evolution equations. The final section discusses the application of the result to the 2d Navier-Stokes equations, illustrating the techniques available for equations that possess a global attractor.

2 Lipschitz ODEs

In this section we give a simple proof of the result for ODEs, following ideas in Kukavica [5]. As well as being more straightforward than the proof of Yorke [12], this also serves as a taster for the more involved argument in the next section.

Theorem 2.1 *Any periodic orbit of the equation $\dot{x} = f(x)$, where f has Lipschitz constant L , has period $T \geq 1/L$.*

As remarked in the introduction, Yorke [12] showed that the period is in fact bounded below by $2\pi/L$.

Proof. Fix $\tau > 0$ and set $v(t) = x(t) - x(t - \tau)$. Then

$$v(t) - v(s) = \int_s^t \dot{v}(r) \, dr.$$

Integrating both sides with respect to s from 0 to T gives

$$Tv(t) = \int_0^T \left(\int_s^t \dot{v}(r) \, dr \right) \, ds$$

and so

$$T|v(t)| \leq \int_0^T \int_0^T |\dot{v}(r)| \, dr \, ds \leq T \int_0^T |\dot{v}(r)| \, dr,$$

i.e.

$$\begin{aligned} |x(t) - x(t - \tau)| &\leq \int_0^T |\dot{v}(s)| \, ds = \int_0^T |f(x(s)) - f(x(s - \tau))| \, ds \\ &\leq L \int_0^T |x(s) - x(s - \tau)| \, ds. \end{aligned}$$

Therefore

$$\int_0^T |x(t) - x(t - \tau)| \, dt \leq LT \int_0^T |x(s) - x(s - \tau)| \, ds,$$

and it follows that if $LT < 1$ then

$$\int_0^T |x(t) - x(t - \tau)| \, dt = 0.$$

Thus $x(t) = x(t - \tau)$ for all $\tau > 0$, i.e. $x(t)$ is constant. \square

We note here that it is clear from the proof that f need only have Lipschitz constant L ‘on the periodic orbit’, i.e.

$$|f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y \in \Gamma,$$

where Γ is the periodic orbit under consideration. In particular this means that the result applies to equations where f is only locally Lipschitz when there exists a bounded attracting set.

3 Lipschitz semilinear evolution equations

We now prove a similar result in an infinite-dimensional setting.

Let H a Hilbert space H , with norm $|\cdot|$ and inner product (\cdot, \cdot) , and let A be an unbounded positive linear self-adjoint operator with compact inverse that acts on H . This means, in particular, that A has a set of orthonormal eigenfunctions $\{w_j\}_{j=1}^\infty$ with corresponding positive eigenvalues λ_j , $Aw_j = \lambda_j w_j$, which form a basis for H .

We denote by $D(A^\alpha)$ the domain in H of the fractional power A^α , which in this setting has the simple characterisation

$$D(A^\alpha) = \left\{ \sum_{j=1}^{\infty} c_j w_j : \sum_{j=1}^{\infty} \lambda_j^{2\alpha} |c_j|^2 < \infty \right\}.$$

Following Henry [4] we consider semilinear evolution equations of the form

$$du/dt = -Au + f(u), \quad (1)$$

where $f(u)$ is locally Lipschitz from $D(A^\alpha)$ into H . There are extensive existence and uniqueness results available for such equations for all $0 \leq \alpha < 1$; in particular solutions are given by the variation of constants formula

$$u(t) = e^{-At} u_0 + \int_0^t e^{-A(t-s)} f(u(s)) ds. \quad (2)$$

In what follows we have to restrict to the case $0 \leq \alpha \leq 1/2$.

Theorem 3.1 *For each α with $0 \leq \alpha \leq 1/2$ there exists a constant K_α such that if*

$$|f(u) - f(v)| \leq L|A^\alpha(u - v)| \quad \text{for all } u, v \in D(A^\alpha)$$

then any periodic orbit of (1) must have period at least $K_\alpha L^{-1/(1-\alpha)}$.

Proof. On a periodic orbit of period T we have

$$u(t) = u(t + T) = e^{-AT} u(t) + \int_0^T e^{-A(T-s)} f(u(s + t)) ds,$$

and so

$$(I - e^{-AT})u(t) = \int_0^T e^{-A(T-s)} f(u(s + t)) ds.$$

It follows that

$$u(t) - u(t + \tau) = (I - e^{-AT})^{-1} \int_0^T e^{-A(T-s)} [f(u(t + s)) - f(u(t + \tau + s))] ds. \quad (3)$$

Since u is T -periodic,

$$\int_0^T f(u(s + t)) ds = \int_0^T f(u(s + t + \tau)) ds,$$

and so in fact for any constant c

$$\begin{aligned} & (I - e^{-AT})(u(t) - u(t + \tau)) \\ &= \int_0^T (e^{-A(T-s)} - cI)(f(u(s+t)) - f(u(s+t+\tau))) \, ds. \end{aligned}$$

Therefore

$$\begin{aligned} & u(t) - u(t + \tau) \\ &= \int_0^T [(I - e^{-AT})^{-1}(e^{-A(T-s)} - cI)] (f(u(s+t)) - f(u(s+t+\tau))) \, ds. \end{aligned}$$

For ease of notation we now write

$$D(t) = u(t) - u(t + \tau) \quad \text{and} \quad F(t) = f(u(t)) - f(u(t + \tau)).$$

Then since the eigenfunctions of A are also the eigenfunctions of

$$(I - e^{-AT})^{-1}(e^{-A(T-s)} - cI),$$

we have, for each $k \in \mathbb{N}$,

$$(A^\alpha D(t), w_k) = \int_0^T \lambda_k^\alpha \frac{e^{-\lambda_k(T-s)} - c}{1 - e^{-\lambda_k T}} (F(t+s), w_k) \, ds,$$

and so

$$|(A^\alpha D(t), w_k)| \leq \frac{\lambda_k^\alpha}{1 - e^{-\lambda_k T}} \left(\int_0^T (e^{-\lambda_k s} - c)^2 \, ds \right)^{1/2} \left(\int_0^T (F(t+s), w_k)^2 \, ds \right)^{1/2}.$$

We now choose $c = (1 - e^{-\lambda_k T})/\lambda_k T$ in order to minimise the first integral, for which we then obtain

$$\int_0^T (e^{-\lambda_k s} - c)^2 \, ds = T \left[\frac{1 - e^{-2\lambda_k T}}{2\lambda_k T} - \frac{(1 - e^{-\lambda_k T})^2}{(\lambda_k T)^2} \right].$$

Therefore

$$|(A^\alpha D(t), w_k)| \leq T^{1/2-\alpha} \Phi(\lambda_k T) \left(\int_0^T (F(t+s), w_k)^2 \, ds \right)^{1/2},$$

where

$$\Phi(\mu) := \frac{\mu^\alpha}{1 - e^{-\mu}} \left[\frac{1 - e^{-2\mu}}{2\mu} - \frac{(1 - e^{-\mu})^2}{\mu^2} \right]^{1/2}.$$

Now, $\Phi(\mu)$ is bounded on $[0, \infty)$ by some constant C_α : it is clear that $\Phi(\mu) \sim \mu^{\alpha-1/2}/\sqrt{2}$ as $\mu \rightarrow \infty$, while a careful Taylor expansion shows that $\Phi(\mu) \sim \mu^\alpha/2\sqrt{3}$ as $\mu \rightarrow 0$.

It follows that for each $k \in \mathbb{N}$

$$|(A^\alpha D(t), w_k)|^2 \leq C_\alpha^2 T^{1-2\alpha} \int_0^T |(F(t+s), w_k)|^2 ds.$$

Summing both sides over all k we obtain

$$|A^\alpha D(t)|^2 \leq C_\alpha^2 T^{1-2\alpha} \int_0^T |F(t+s)|^2 ds \leq C_\alpha^2 T^{1-2\alpha} L^2 \int_0^T |A^\alpha D(s)|^2 ds.$$

Now integrate the left- and right-hand sides of this expression with respect to t between $t = 0$ and $t = T$ to obtain

$$\int_0^T |A^\alpha D(t)|^2 dt \leq C_\alpha^2 T^{2-2\alpha} L^2 \int_0^T |A^\alpha D(s)|^2 ds.$$

Therefore if $C_\alpha T^{1-\alpha} L < 1$ we must have

$$\int_0^T |A^\alpha(u(t) - u(t+\tau))|^2 dt = 0.$$

It follows that $u(t) = u(t+\tau)$ for all $t \in [0, T]$, and since this holds for any $\tau > 0$, $u(t)$ must be a constant orbit. Therefore any periodic orbit must have period at least $K_\alpha L^{1/(1-\alpha)}$. \square

Note that the proof essentially consists of obtaining a bound on the norm of the mapping

$$f \mapsto \int_0^T (I - e^{-AT})^{-1} e^{-As} f(s+t) ds$$

as an operator from $\dot{L}_{\text{per}}^2(0, T; H)$ into $L^2(0, T; D(A^\alpha))$, where

$$\begin{aligned} \dot{L}_{\text{per}}^2(0, T; H) = \{f : f|_{[0, T]} \in L^2(0, T; H), \int_0^T f(s) ds = 0, \\ f(t+T) = f(t) \text{ a.e. } t \in \mathbb{R}\} \end{aligned}$$

endowed with the standard norm on $L^2(0, T; H)$. Although this seems to shed little light on the proof, it should be possible to extend the method by proving a similar bound in a more general situation, for example A is a sectorial operator, as treated by Henry [4].

4 Application to the 2d Navier-Stokes equations

As an example we now consider 2d Navier-Stokes equations, with periodic boundary conditions. The bound on the period that follows from Proposition 3.1 has already been obtained by Kukavica [5], but the argument here is intended to illustrate the application of our result to a concrete problem.

The main point of this example is that although we have to take α strictly greater than one half to ensure that the nonlinear term is Lipschitz from $D(A^\alpha)$ into H (and even then only *locally* Lipschitz), by using the fact that any periodic orbit must be contained in the global attractor, which is a bounded subset of H^2 , we can show that *on the attractor* the nonlinear term is Lipschitz from $D(A^{1/2})$ into H , and thus apply Theorem 3.1.

Initially we consider the 2d Navier-Stokes equations

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = h \quad \nabla \cdot u = 0$$

on $Q = [0, l]^2$ with periodic boundary conditions on Q ($u(x + le_j, t) = u(x, t)$) and zero total momentum ($\int_Q u = \int_Q h = 0$). Since we will want to keep careful track of the dependence of the minimal period on ν and l , it is convenient to rescale the variables to put the equation into non-dimensionalised form.

To this end we set

$$\tilde{u} = lu/\nu, \quad \tilde{x} = x/l, \quad \tilde{t} = \nu t/l^2, \quad \tilde{p} = l^2 p/\nu^2, \quad \text{and} \quad \tilde{h} = l^3 h/\nu^2$$

and so obtain

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} - \Delta_{\tilde{x}} \tilde{u} + (\tilde{u} \cdot \nabla_{\tilde{x}}) \tilde{u} + \nabla_{\tilde{x}} \tilde{p} = \tilde{h} \quad \nabla_{\tilde{x}} \cdot \tilde{u} = 0$$

on the new domain $\tilde{Q} = [0, 1]^2$.

We drop the tildes, and via standard manipulations rewrite these equations as an evolution equation on

$$H = \left\{ u \in L^2(Q) : \nabla \cdot u = 0, \int_Q u = 0 \right\},$$

namely

$$\frac{du}{dt} + \nu Au + B(u, u) = g, \quad (4)$$

where A is the Stokes operator $A = -\Pi\Delta$, $B(u, u) = \Pi(u \cdot \nabla)u$, and $g = \Pi\tilde{h}$, where Π is the orthogonal projection from L^2 onto H (see Henry [4] or Temam [11] for details).

The fractional power space $D(A^\alpha)$ is a subset of $H^{2\alpha}(Q)$ (see Constantin & Foias [2], for example), and for $u \in D(A^\alpha)$ the norms are equivalent,

$$c_\alpha |A^\alpha u| \leq \|u\|_{H^{2\alpha}} \leq C_\alpha |A^\alpha u| \quad c_\alpha \leq 1 \leq C_\alpha.$$

It follows from standard Sobolev embedding theorems that B is Lipschitz from $D(A^\alpha)$ into H for $\alpha > 1/2$. Indeed, using the bilinearity of B ,

$$|B(u, u) - B(v, v)| = |B(u - v, u) + B(v, u - v)| \leq |B(u - v, u)| + |B(v, u - v)|,$$

from whence

$$\begin{aligned} |B(u, u) - B(v, v)| &\leq |u - v|_{L^p} |Du|_{L^q} + \|v\|_{L^\infty} |D(u - v)|_{L^2} \\ &\leq C_p |D(u - v)|_{L^2} \|Du\|_{H^{1-(2/q)}} + C_\epsilon \|v\|_{H^{1+\epsilon}} |D(u - v)|_{L^2} \\ &\leq C_\alpha \left[|D(u - v)|_{L^2} |A^\alpha u| + |A^\alpha v| |D(u - v)|_{L^2} \right] \\ &\leq C_\alpha \left[|A^\alpha u| + |A^\alpha v| \right] |A^\alpha(u - v)|, \end{aligned}$$

choosing $1 - (1/q) = \alpha$ and $1 + \epsilon = 2\alpha$.

However, when $g \in H$ then the equation possesses a global attractor \mathcal{A} . This attractor, which contains all the periodic orbits, is a bounded subset of $D(A) \subset H^2(\Omega)$. We can use the greater regularity of functions on the attractor to show that there B is Lipschitz from $D(A^{1/2})$ into H .

Defining the Grashof number by $G = |g|_{L^2}$ (this agrees with the Grashof number for the dimensional model, which is usually defined by $G = l^2|h|_{L^2(Q)}/\nu^2$), we have the bounds

$$|u| \leq cG, \quad |Du| \leq cG, \quad \text{and} \quad |Au| \leq cG(1 + G^2), \quad (5)$$

for all $u \in \mathcal{A}$. See, for example, Robinson [7], which in particular includes (albeit less explicitly) the asymptotic estimate on $|Au|$. Sharper estimates are available if one is prepared to take into account the norm of g in H^1 , see Kukavica [6].

We now need to estimate the Lipschitz constant of $f(u) := -B(u, u) + g$ on the attractor. To this end, observe that

$$f(u) - f(v) = (-B(u, u) + g) - (-B(v, v) + g) = -B(w, u) - B(v, w),$$

where $w = u - v$. Clearly $|B(v, w)| \leq \|v\|_\infty |Dw|$, and using the inequality

$$\|\phi\|_\infty \leq c|D\phi| \left[1 + \log \frac{|A\phi|^2}{|D\phi|^2} \right]^{1/2}$$

(due to Brézis & Gallouet [1]) we can estimate

$$|B(v, w)| \leq c|Dw||Dv| \left[1 + \log \frac{|Av|^2}{|Dv|^2} \right]^{1/2}.$$

We can obtain a very similar estimate for the other term but we need to be a little more careful. If $|Au| \leq 2|Du|$ then we can estimate directly

$$|B(w, u)| \leq \|w\|_{L^4} \|Du\|_{L^4} \leq c|Dw||Au| \leq c|Dw||Du| \leq cG|Dw|.$$

However, if $|Au| > 2|Du|$ then the argument is more involved. Noting the dependence of the constant in the Sobolev embedding $H^1(\Omega) \subset L^p(\Omega)$,

$$\|u\|_{L^p} \leq cp^{1/2}|Du|, \tag{6}$$

(see Talenti [9]), we can use the Lebesgue interpolation inequality

$$\|\phi\|_{L^{2+\epsilon}} \leq |\phi|^{1-\epsilon} \|\phi\|_{L^{2(2+\epsilon)/(1+\epsilon)}}^\epsilon \quad (0 \leq \epsilon < 1)$$

to deduce that

$$\|\phi\|_{L^{2+\epsilon}} \leq c|\phi|^{1-\epsilon}|D\phi|^\epsilon.$$

It follows that

$$\begin{aligned} |B(w, u)| &\leq \|w\|_{L^{2(2+\epsilon)/\epsilon}} \|Du\|_{L^{2+\epsilon}} \\ &\leq c \left(\frac{2+\epsilon}{\epsilon} \right)^{1/2} |Dw||Du|^{1-\epsilon} |Au|^\epsilon \\ &= c \left(1 + \frac{2}{\epsilon} \right)^{1/2} \left(\frac{|Au|}{|Du|} \right)^\epsilon |Du||Dw|. \end{aligned}$$

Now choose $\epsilon = 2/\log(|Au|/|Du|)$ (since $|Au| > 2|Du|$ we have $0 \leq \epsilon < 1$) so that

$$|B(w, u)| \leq c \left(1 + \log \frac{|Au|}{|Du|} \right)^{1/2} |Du||Dw|.$$

Now for $|Du| \geq 1$ we have

$$|Du|^2 \left(1 + \log \frac{|Au|}{|Du|} \right) \leq |Du|^2 (1 + \log |Au|) \leq cG^2(1 + \log G),$$

while for $|Du| < 1$ the upper bound can be rewritten as

$$\begin{aligned} & |Du|^2 (1 + \log |Au|) - |Du|^2 \log |Du| \\ & \leq |Du|^2 (1 + \log |Au|) + c|Du| \\ & \leq cG^2(1 + \log G) + cG. \end{aligned}$$

Combining all these estimates we therefore obtain, for $G \geq 1$,

$$|f(u) - f(v)| \leq cG(1 + \log G)^{1/2} |D(u - v)|.$$

Applying proposition 3.1 to (4), where the right-hand side has Lipschitz constant $L = cG(1 + \log G)^{1/2}$ and $\alpha = 1/2$ we can deduce that the minimal period *using the rescaled time* is at least $cG^{-2}(1 + \log G)^{-1}$. Returning to the original timescale the period is at least $c\nu^{-1}l^2G^{-2}(1 + \log G)^{-1}$, as found by Kukavica [5].

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